Axisymmetric thin disk model

Accretion disk configuration around a central object.

Features

- *stationary configuration*
- *compact*: $M_S \sim 1.5 - 2M_\odot$
  $R_S \sim R_{Sch} = 2GM_S/c^2$
- *strongly magnetized*: $B \sim 10^{12} - 10^{14}$G

Thin disk condition: $H_0(r)/r \ll 1$

Newton potential: $$\chi(r,z) = -\frac{GM_S}{\sqrt{r^2+z^2}}$$

Axisymmetric Field: $$\vec{B} = -\frac{1}{r}\partial_z\psi\vec{e}_r + \frac{I}{r}\vec{e}_\phi + \frac{1}{r}\partial_r\psi\vec{e}_z$$

Dipole Field approximation: $$\psi_D = \frac{\mu_0 r^2}{(r^2+z^2)^{3/2}}$$

Corotation theorem: $\omega = \omega(\psi)$

$\rightarrow H_0$: half depth of the disk.

Fluid Dynamics

- Radial Eq: $$\rho_0 v_r \frac{\partial v_r}{\partial r} - \rho_0 \left(\omega^2 r - \frac{\partial \chi}{\partial r} \right) + \frac{\partial p}{\partial r} = 0$$

- Vertical Eq: $$\frac{\partial p}{\partial z} = -\rho \frac{\partial \chi}{\partial z}$$

- Azimuthal Eq: $$\frac{\dot{M}}{2\pi r} \frac{\partial j}{\partial r} = \frac{2}{3} \alpha \frac{\partial}{\partial r} \left(\Sigma v_{s0} H_0 r^3 \frac{\partial \omega}{\partial r} \right)$$

- Continuity Eq: $$\frac{\partial \dot{M}}{\partial r} = 0$$

Definitions:

$$\Sigma = \int_{-z_0}^{z_0} \rho dz \qquad \dot{M} = -2\pi r \Sigma v_r \qquad j = r v_\phi$$

$$\rho_0 = \rho(z = 0) \qquad \omega = \frac{v_\phi}{r}$$

[G.S. Bisnovatyi-Kogan, R.V.E. Lovelace, *New. Astro. Rev.* **45**, 663 (2001)]

In the case of a polytropic equation of state

$$p = \kappa \rho^{1+1/\gamma}$$

from the vertical equation we have that

$$\rho = \rho_0 \left(1 - \frac{z^2}{H_0^2}\right)^\gamma = \left[\frac{GM_S}{2\kappa(\gamma+1)}\right]^\gamma \left(1 - \frac{z^2}{H_0^2}\right)^\gamma$$

Isothermal case: $p = \kappa\rho$

$$\rho = \rho_0 \exp\left(-\frac{z^2}{H_0^2}\right) \qquad H_0 = \left(\frac{2\kappa r^3}{GM_S}\right)^{1/2}$$

$$\Sigma = \left(\frac{2\pi\kappa}{GM_S}\right)\rho_0 r^{3/2} \qquad v_{s0} = \sqrt{\kappa}$$

$$\omega \simeq \omega_K = \sqrt{\frac{GM_S}{r^3}}$$

From the azimuthal eq, introducing the coefficient α,

$$\frac{\dot{M}}{2\pi}(j - j_0) = \frac{2}{3}\alpha\Sigma v_{s0} H_0 r^3 \frac{\partial\omega}{\partial r}$$

Positive value of j_0 corresponds to a negative total flux
$\Rightarrow$ The central body accretes its total angular momentum.

Ngative value of j_0 corresponds to a mass increase
$\Rightarrow$ Te central body decreases its total angular momentum.

Shakura Standard Model and its issues

The inward matter flux is due to viscous stresses.
The coefficient α scales the viscosity coefficient η_v:

$$\eta_v = \frac{2}{3}\alpha\rho v_{s0} H_0 \iff \dot{M} = 4\pi\alpha\rho v_{s0} H_0^2 \iff \tau_{r\phi} = -\alpha\rho v_{s0}^2$$

Turbulent enhancement: a non-vanishing viscosity exists in quasi-ideal astrophysical plasmas, but the Shakura paradigm hides in α every *effective* treatment needed to reach the observed accretion rate.

Let $\xi < 1$ be an efficiency such that the luminosity $\mathcal{L}_{\text{acc}} = \xi\mathcal{L}_{\text{Edd}}$:

$$\frac{GM_{\text{S}}\dot{M}_\alpha}{R_{\text{S}}} = \xi\frac{GM_{\text{S}}m_p c}{\sigma_{\text{Th}}} \Longrightarrow \alpha_{\text{Sh}} = \xi\frac{c}{v_{s0}}\frac{R_{\text{S}}}{H_0}\frac{\ell_{\text{Th}}}{H_0}$$

where σ_{Th} is the Thomson cross-section. The microscopical viscosity due to ion-ion collisions is estimated as:

$$\eta_v = \frac{\rho v_{s0}^2}{\nu_{ii}} \Longrightarrow \alpha_{\text{id}} = \frac{3}{2}\frac{\ell_{\text{ii}}}{H_0}$$

The *effective* parametrization accounts for several orders of magnitude with respect to microscopical conditions:

$$\frac{\alpha_{\text{Sh}}}{\alpha_{\text{id}}} = \frac{2}{3}\xi\frac{c}{v_{s0}}\frac{R_{\text{S}}}{H_0}\frac{\ell_{\text{Th}}}{\ell_{\text{ii}}} \simeq 10^{-3}\cdot 10^{3}\cdot 10^{-2}\cdot 10^{10} = 10^{8}$$

Towards an alternative paradigm

Generalized Ohm Law: background $\vec{B}_0$ and backreaction $\vec{B}_1$

$$\vec{E}_\phi + \frac{1}{c}(\vec{v} \times \vec{B})_\phi = \vec{J}_\phi/\sigma \quad \Longrightarrow \quad v_r B_0 \simeq \frac{c^2}{4\pi}\frac{1}{\sigma}\frac{B_1}{\lambda}$$

λ is the backreaction length-scale, responsible for the induced currents.

The **Magnetic Prandtl Number**: since v_r depends on radius r and viscosity η_v, we can derive:

$$\mathrm{PrM}(n_e, T) \equiv \frac{4\pi\sigma\eta_v}{\rho c^2} \approx \frac{r}{\lambda}\frac{B_1}{B_0}$$

It can be shown that $\mathrm{PrM} \gg 1$ in typical disk ranges

- $10^8 \lesssim n_e \lesssim 10^{12}\mathrm{cm}^{-3}$
- $10^5 \lesssim T \lesssim 10^8\mathrm{K}$

In the Standard Model, $B_1 \ll B_0$ and $\lambda \simeq r \simeq R_{\rm disk}$:
effective small PrM with a surprisingly small effective conductivity.

For reasonable $B_1 \lesssim B_0$, the huge values of realistic (i.e., non-effective) PrM ask for the formation of **microstructures** with $\lambda \ll r \simeq R_{\rm disk}$.

[G. Montani, J. Petitta, *Phys. Rev. E* **87**, 053111 (2013)]

Local Formulation

- We choose a **fixed value** $r = r_0$
- Isothermal condition
- Hierarchy ordering of the gradients

$$\psi = \psi_0 + \psi_1 \quad (\psi_1 \ll \psi_0)$$

$$\omega \simeq \omega_K + \delta\omega \simeq \omega_K + \psi_1 d\omega/d\psi_0$$

$$\rho = \bar{\rho}(r_0, z^2) + \hat{\rho}(r_0, z^2, r - r_0)$$

$$p = \bar{p}(r_0, z^2) + \hat{p}(r_0, z^2, r - r_0)$$

Vertical Configuration

$$\begin{cases} D(z^2) \equiv \dfrac{\bar{\rho}}{\rho_0(r_0)} = \exp\left(-\dfrac{z^2}{H_0^2}\right) \\ \rho_0(r_0) \equiv \rho(r_0,\, 0)\,,\; H_0^2 \equiv \dfrac{4K_B\bar{T}}{m_i\omega_K^2} \\ \partial_z\hat{p} + \omega_K^2 z\hat{\rho} + \dfrac{1}{4\pi r_0^2}\left(\partial_z^2\psi_1 + \partial_r^2\psi_1\right)\partial_z\psi_1 = 0 \end{cases}$$

Radial Configuration

$$\begin{cases} \omega \simeq \omega_K + \delta\omega \simeq \omega_0(\psi_0) + \omega_0'\psi_1 \\ 2\omega_K r_0(\bar{\rho} + \hat{\rho})\omega_0'\psi_1 + \dfrac{1}{4\pi r_0^2}\left(\partial_z^2\psi_1 + \partial_r^2\psi_1\right)\partial_{r_0}\psi_0 = \\ = \partial_r\left[\hat{p} + \dfrac{1}{8\pi r_0^2}\left(\partial_r\psi_1\right)^2\right] + \dfrac{1}{4\pi r_0^2}\partial_r\psi_1\partial_z^2\psi_1 \end{cases}$$

[B. Coppi, *Phys. Plasmas* **12**, 057302 (2005)]