Asymptotic methods applied to plasma physics

A.Yu.Anikin* S. Yu. Dobrokhotov †, A. I. Klevin ‡
, Brunello Tirozzi§

Аннотация

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1 Introduction

Problem of pencil of operators $\widehat{\mathbf{H}}(\hat{x}, \hat{p}, E)\Psi(E) = 0$, in the case of operator $\widehat{\mathbf{H}} = \widehat{\mathbf{G}}(\hat{x}, \hat{p}) - E$ it is the spectral problem.

Asymptotic solution of this problem: the number E and the function Ψ (with the normalization condition $\|\Psi\|_{L^2} = 1$) such that $\|\widehat{\mathbf{H}}\Psi\|_{L^2} = O(h^\beta)$, where $\beta > 0$. E is the asymptotic eigenvalue and $\Psi(E)$ is the asymptotic eigenfunction

1.1 Asymptotic in the vector case

Let us consider a $N \times N$ matrix operator

$$\hat{\mathbf{H}}(E)\Psi = 0, \qquad \mathbf{H} = \mathbf{H}_0 + h\mathbf{H}_1 + \dots .$$

Solve

$$\mathbf{H}$$

*Institut of Problem of Mechanics A. Yu. Ishlinsky RAS and Moscow Physics Technical Institut, Mosco Technical University Bauman, E-mail: anikin83@inbox.ru

†Institut of Problem of Mechanics A. Yu. Ishlinsky RAS and Moscow Physics Technical Institut, Mosco Technical University Bauman, E-mail: dobr@ipmnet.ru

‡Institut of Problem of Mechanics A. Yu. Ishlinsky RAS and Moscow Physics Technical Institut, Moscow Technical University Bauman E-mail: klyovin@mail.ru

§Laboratori Enea Frascati, Department of Physics Rome University Italy,E-mail: brunellotirozzi@gmail.com

using the scalar hamiltonian

$$\mathcal{H} = \det \mathbf{H}_0$$

Equivalence: Principle of Maupertuis-Jacobi.

The formula for the asymptotic solution of the problem is derived by by these assumptions

- The variable p_ϕ is a first integral of the Hamiltonian system $\mathcal{H} = \mathcal{H}(r, \theta, p_r, p_\theta, p_\phi, E)$
- For all $p_\phi \in (p_\phi^1, p_\phi^2)$ and $E = E(p_\phi)$, a non degenerate minimum $z_0 = (r_0(p_\phi, \theta_0(p_\phi), p_{r0} = 0, p_\theta = 0)$ exists such that $\mathcal{H}(r, \theta, p_r, p_\theta, p_\phi, E(p_\phi) = 0$
- $\frac{\partial \mathcal{H}}{\partial E} \neq 0$, $\frac{\partial \mathcal{H}}{\partial p_\phi} \neq 0$

Theorem 1. *Suppose that the above condition are satisfied and that $p_\phi = p_\phi^0$ satisfies this condition (Bohr-Sommerfeld quantum condition), :*

$$p_\phi^0 = h\left(k + \frac{(b_1 + b_2)T_s}{4\pi}\right), \qquad k \in \mathbb{Z}.$$

That there is a periodic solution $X_0(t), P_0(t)$ with period T_s of the Hamilton equations generated by $\mathcal{H}$ and b_1, b_2 are eigenvalues of the variation equations around the periodic orbit. Let $E = E(p_\phi^0) + h\lambda_0$, were λ_0 is computed by some other asymptotic formula . Let $dS_0(X_0(s), P_0(s)) = X_0 dP_0$. Then the eigenfunction $\psi(x)$ of the full $N \times N$ problem $\widehat{\mathbf{H}}(\hat{x}, \hat{p}, E)\Psi(E) = 0$ has the asymptotic representation

$$\psi(x) = \exp \frac{i}{h}\left(S_0(s) + \frac{2\pi s}{T_s}(kh - p_\phi^0) + \langle P_0(s), x - X_0(s)\rangle + \frac{1}{2}\left\langle \mathcal{BC}^{-1}(x - X_0(s)), x - X_0(s)\right\rangle\right),$$

where $\mathcal{BC}^{-1}$ are the solutions of the variation equation around the periodic orbit and $s = s(x)$ is determined everywhere by the condition $< x - X_0(s), \dot{X}_0(s) >= 0$

2 Application to Neutron Star's Magnetosphere

2.1 Equations

Consider a gas of electrons and positrons with density $n = 10^{12}/cm^3$, this is the critical Goldreich Julian density on the star surface which has the meaning of the density to screen the longitudinal electric field, $m_i = m_e = 9.1 \times 10^{-28}\text{gms} = m$, $q_i = -q_e = (4.8) \times 10^{-10}\text{esu}$ The linearized system of equations

$$\begin{array}{c} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mathbf{c}^2 \nabla \wedge \nabla \wedge \mathbf{E} + \frac{nq^2}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_1 \wedge \mathbf{B}) + \frac{nq^2}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_2 \wedge \mathbf{B}) = 0, \\ \frac{\partial \mathbf{v}_1}{\partial t} - \frac{q}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_1 \wedge \mathbf{B}) = 0, \\ \frac{\partial \mathbf{v}_2}{\partial t} + \frac{q}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_2 \wedge \mathbf{B}) = 0. \end{array} \tag{1}$$

Let us introduce a perturbative parameter h and make the change of variables $x = \mathbf{c}x'/h$, $t = t'/h$ we write t instead of t'.

Let us introduce another vector function $\mathbf{W} = -ih\frac{\partial \mathbf{E}}{\partial t}$ such that we can rewrite (1) in the form

$$\begin{array}{c} ih\frac{\partial \mathbf{E}}{\partial t} = -\mathbf{W} \\ ih\frac{\partial \mathbf{W}}{\partial t} = (-ih\nabla) \wedge (-ih\nabla) \wedge \mathbf{E} - \frac{nq^2}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_1 \wedge \mathbf{B}) - \frac{nq^2}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_2 \wedge \mathbf{B}), \\ ih\frac{\partial \mathbf{v}_1}{\partial t} = i\frac{q}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_1 \wedge \mathbf{B}), \\ ih\frac{\partial \mathbf{v}_2}{\partial t} = -i\frac{q}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_2 \wedge \mathbf{B}). \end{array}$$

Let us introduce $\Phi = (\mathbf{E}, \mathbf{W}, \mathbf{v}_1, \mathbf{v}_2)^T$ a vector with 12 components. Then the system has the form:

$$ih\frac{\partial \Phi}{\partial t} = \hat{\mathbf{A}}\Phi, \quad \hat{\mathbf{A}} = \mathbf{A}(x, -ih\nabla, h) = \mathbf{A}_0(x, -ih\nabla) + O(h^2)$$

where $\mathbf{A}(x, p)$ – is the symbol of the operator $\hat{\mathbf{A}}$:

$$\mathbf{A}(x, \hat{p})\Phi = \begin{pmatrix} -\mathbf{W} \\ (-ih\nabla) \wedge (-ih\nabla) \wedge \mathbf{E} - \frac{nq^2}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_1 \wedge \mathbf{B}) - -\frac{nq^2}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_2 \wedge \mathbf{B}) \\ i\frac{q}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_1 \wedge \mathbf{B}) \\ -i\frac{q}{m}(\mathbf{E} + \frac{1}{\mathbf{c}}\mathbf{v}_2 \wedge \mathbf{B}) \end{pmatrix}.$$

We look for the stationary solution setting

$$\Phi = e^{-\frac{i\omega t}{h}}\Psi(x) \equiv e^{-\frac{i\omega t}{h}}(\mathbf{W}(x), \mathbf{E}(x), \mathbf{v}_1(x), \mathbf{v}_2(x))^T.$$

Omitting the primes and putting $\hat{p} = -ih\nabla$ and $\omega^2 = \frac{nq^2}{m}$

$$\begin{pmatrix} -\mathbf{W} \\ \hat{p}\wedge\hat{p}\wedge\mathbf{E} - \omega^2(\mathbf{E} - \frac{1}{\mathbf{c}}\mathbf{B}\wedge\mathbf{v}_1) - \omega^2(\mathbf{E} - \frac{1}{\mathbf{c}}\mathbf{B}\wedge\mathbf{v}_2) \\ i\frac{q}{m}(\mathbf{E} - \frac{1}{\mathbf{c}}\mathbf{B}\wedge\mathbf{v}_1) \\ -i\frac{q}{m}(\mathbf{E} - \frac{1}{\mathbf{c}}\mathbf{B}\wedge\mathbf{v}_2) \end{pmatrix} = \omega \begin{pmatrix} \mathbf{E} \\ \mathbf{W} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}. \qquad (2)$$

2.2 Explicit form of the magnetic field and $\mathcal{H}$

We introduce toroidal coordinates for the magnetosphere

$$\begin{aligned} x &= (a + r\cos\theta)\cos\phi \\ y &= (a + r\cos\theta)\sin\phi \\ z &= r\sin\theta \end{aligned}$$

where $0 < r < 10^{10} - 10^6 = R$, $a = 10^6$cm is the radius of the neutron star $R \sim 10^{10}$ cm is the extension of the magnetosphere taken to be given by the light surface. This surface is defined as the surface where the electric field equals the magnetic field $|\mathbf{E}| = |\mathbf{H}|$.

$$\mathbf{B} = \begin{pmatrix} B_r(r,\theta) \\ B_\theta(r,\theta) \\ B_\phi \end{pmatrix} \equiv \frac{|\mathbf{m}|}{r^3}\begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

where $\mathbf{m}$ is the norm of the magnetic dipole of the star $|\mathbf{m}| = \frac{B_0 a^3}{2} = \frac{10^{12}10^{18}}{2} = \frac{10^{30}}{2}$, $B_0 = 10^{12}$ Gauss is the typical magnetic field of neutron stars.

The other physical variables

$$\mathbf{P}^2 = p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{(a + r\cos\theta)^2},$$
$$\mathbf{B}^2 = \frac{|\mathbf{m}|^2}{r^6},$$
$$\langle \mathbf{P}, \mathbf{B} \rangle = \frac{|\mathbf{m}|}{r^3}\Big(p_r \sin\theta + p_\theta \cos\theta\Big)$$

With these notations we have the following form of $\mathcal{H}$

Introduce the notation, $q_p = -q_e$ is the charge of the positron

$$\Omega = \Omega_1 = \frac{q_e}{m_e c} = -\frac{q}{m_e c}, \quad \Omega_2 = -\Omega_1 = \frac{q_p}{m_p c}$$

$$\alpha = 2\omega_1^2 \Omega^2,$$

$$\gamma = \omega_1^4 (2\Omega_2^2).$$

Where $\omega_1^2 = \frac{nq^2}{m}$ is the plasma frequency and we put also $\omega_{\mathrm{pl}}^2 = 2\omega_1^2$

$$\begin{aligned} \mathcal{H}(p, x, \omega) = \omega^2 \mathbf{P}^2 \mathbf{B}^2 \langle \mathbf{P}, \mathbf{B} \rangle^2 (\omega_{\mathrm{pl}}^2) \Omega^4 \\ -\omega^4 \Big[\langle \mathbf{P}, \mathbf{B} \rangle^2 \mathbf{P}^2 \alpha + \mathbf{B}^2 \langle \mathbf{P}, \mathbf{B} \rangle^2 (\omega_{\mathrm{pl}}^2) \Omega^4 + \mathbf{B}^2 \mathbf{P}^2 (\gamma) + \\ \mathbf{B}^2 \mathbf{P}^4 \alpha + \mathbf{B}^4 \mathbf{P}^2 (\omega_{\mathrm{pl}}^2) \Omega^4 + \mathbf{B}^4 \mathbf{P}^4 \Omega^4 + \langle \mathbf{P}, \mathbf{B} \rangle^2 \gamma \Big] \\ +\omega^6 \Big[(\mathbf{P}^4 (\omega_{\mathrm{pl}}^2) + (\omega_{\mathrm{pl}}^2)^3 + 2\mathbf{P}^2 (\omega_{\mathrm{pl}}^2)^2 + \mathbf{B}^2 \mathbf{P}^4 \Omega^2 + \mathbf{B}^2 \mathbf{P}^2 (5\alpha) + \\ \langle \mathbf{P}, \mathbf{B} \rangle^2 \alpha + \mathbf{B}^2 (2\gamma) + 2\mathbf{B}^4 \mathbf{P}^2 \Omega^4 + \mathbf{B}^4 (\omega_{pl}^2) \Omega^4 \Big] \\ -\omega^8 \Big[\mathbf{P}^4 + 4\mathbf{P}^2 (\omega_{\mathrm{pl}}^2) + 3(\omega_{\mathrm{pl}}^2)^2 + 2\mathbf{B}^2 \mathbf{P}^2 \Omega^2 + \mathbf{B}^2 (4\alpha) + \mathbf{B}^4 \Omega^4 \Big] \\ +\omega^{10} \Big[2\mathbf{P}^2 + 3(\omega_{\mathrm{pl}}^2) + \mathbf{B}^2 \Omega^2 \Big] - \\ \omega^{12} = 0 \end{aligned} \tag{3}$$

p_ϕ as a first integral and also look for the minima of this hamiltonian for the points $p_r = p_\theta = 0$ we get two different type of graphs according to the range of values of p_r

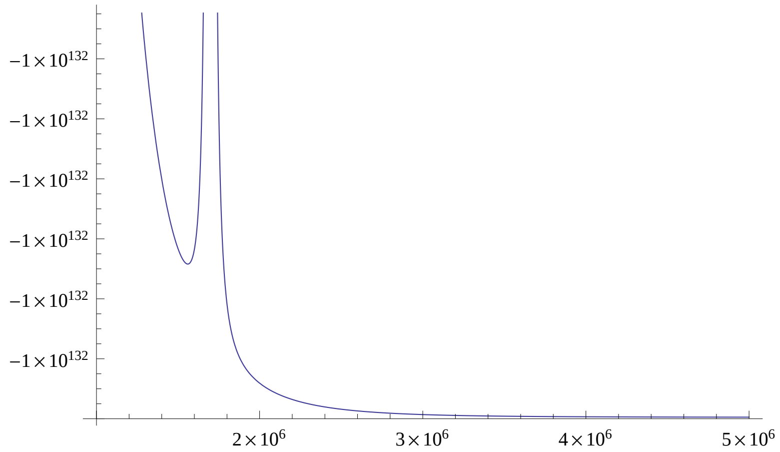

Рис. 1: Graphic of $\mathcal{H}_0(r)$ for $p_r \in (10^9, 6 \times 10^9)$, $\omega = 7.03 \times 10^{10}$.

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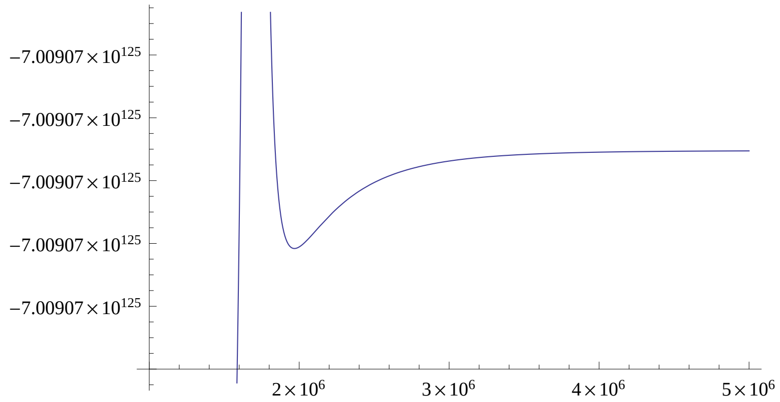

Рис. 2: Graphic of $\mathcal{H}_0(r)$ for $p_r \in (7 \times 10^9, 10^{11})$, $\omega = 7.03 \times 10^{10}$

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